

THE HYPERMETRIC CONE ON 8 VERTICES AND SOME GENERALIZATIONS

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ABSTRACT. The lists of facets – 298,592 in 86 orbits – and of extreme rays – 242,695,427 in 9,003 orbits – of the hypermetric cone HYP_8 are computed. The first generalization considered is the hypermetric polytope HYP_n for which we give general algorithms and a description for $n \leq 8$. Then we shortly consider generalizations to simplices of volume higher than 1, hypermetric on graphs and infinite dimensional hypermetrics.

1. INTRODUCTION

Metric, cut and hypermetric cones are among central objects of Discrete Mathematics. For example, finite metrics and l_1 -metrics can be studied by polyhedral cones or polytopes; see, for example, [15]. The *hypermetric cone* HYP_n is the set of all *hypermetrics on n points*, i.e., the functions $d : \{1, \dots, n\}^2 \rightarrow \mathbb{R}$, such that

$$H(b, d) = \sum_{1 \leq i < j \leq n} b_i b_j d(i, j) \leq 0 \text{ for all } b \in \mathbb{Z}^n, \sum_i b_i = 1.$$

If b is $\{0, \pm 1\}$ -valued and has $k + 1$ ones, the inequality is called $(2k + 1)$ -gonal *inequality*. In fact, the case of general b can be seen as some such $\{0, \pm 1\}$ -valued b on a multiset of $\sum_{i=1}^n |b_i|$ points, in which different points occur $|b_1|, \dots, |b_n|$ times.

The *metric cone* MET_n is the set of all *semimetrics on n points*, i.e., those of above functions d , which satisfy all *triangle* (i.e., all 3-gonal) *inequalities*.

For a set $S \subseteq \{1, \dots, n\}$ the *cut* (or *split*) *semimetric* δ_S is a vector defined as

$$\delta_S(x, y) = \begin{cases} 1 & \text{if } |S \cap \{x, y\}| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\delta_{\bar{S}} = \delta_S$, and δ_S is also can be seen as the adjacency matrix of a *cut* (into S and \bar{S}) *subgraph* of K_n . The *cut cone* CUT_n is the positive span of the $2^{n-1} - 1$ non-zero cut semimetrics; the *cut polytope* $CUTP_n$ is the convex hull of all 2^{n-1} cut semimetrics.

We have the evident inclusions $CUT_n \subseteq HYP_n \subseteq MET_n$ with $CUT_n = MET_n$ only for $3 \leq n \leq 4$; also, $CUT_n = HYP_n$ only for $3 \leq n \leq 6$ ([15]). So, the first proper HYP_n is HYP_7 ; it was described in [13]. While CUT_n is important in Analysis and Combinatorics, the cone HYP_n is related to Voronoi-Delaunay theory in Geometry of Numbers.

Given a polyhedral cone C , let e_C denote the number of extreme rays of C , while f_C denote the number of facets of C . The Table 1 gives e_C and f_C for $C = CUT_n, HYP_n, MET_n$ with $n = 3, 4, 5, 6, 7, 8$. The enumeration of orbits of facets of CUT_n for $n \leq 7$ was done in [32, 2, 23] for $n = 5, 6$ and 7, respectively.

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TABLE 1. The number of extreme rays and facets in cones HYP_n , CUT_n and MET_n for $3 \leq n \leq 8$; the numbers of orbits under the symmetric group $Sym(n)$ are given in parentheses. Also, the number of vertices and facets in the hypermetric polytope $HYPP_n$, see Section 4, is given for $3 \leq n \leq 8$, with number of orbits under $Sym(n)$ and 2^{n-1} switchings

C	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$
CUT_n, e	3(1)	7(2)	15(2)	31(3)	63(3)	127(4)
CUT_n, f	3(1)	12(1)	40(2)	210(4)	38,780(36)	49,604,520(2,169)
HYP_n, e	3(1)	7(2)	15(2)	31(3)	37,170(29)	242,695,427(9,003)
HYP_n, f	3(1)	12(1)	40(2)	210(4)	3,773(14)	298,592(86)
MET_n, e	3(1)	7(2)	25(3)	296(7)	55,226(46)	119,269,588(3,918)
MET_n, f	3(1)	12(1)	30(1)	60(1)	105(1)	168(1)
$HYPP_n, v$	4(1)	8(1)	16(1)	32(1)	113,152(6)	1,388,383,872(581)
$HYPP_n, f$	4(1)	16(1)	56(2)	68(3)	10,396(7)	1,374,560(22)

For CUT_8 and $CUTP_8$, sets of facets were found in [7]; completeness of these sets was shown in [14]. The enumeration of orbits of extreme rays of MET_n for $n \leq 8$ was done in [24, 9, 10].

In Section 2 the facets of the hypermetric cone HYP_8 are determined with the help of the connection with geometry of numbers and the list of simplices of dimension 7. In Section 3 we list the extreme rays of HYP_8 by using the list of extreme Delaunay polytopes in dimension 7.

In Section 4 we define the hypermetric polytope and give algorithms for computing with it. This is then used to compute the vertices and facets of the hypermetric polytope $HYPP_7$ and $HYPP_8$. In Section 5 we give the list of facets of the cone that come from simplices of volume greater than 1. Those are direct analogues of the hypermetric cone. In Section 6 we define an analogue of hypermetric inequality for the cut-polytope of a graph G . This directly generalizes the corresponding definitions of the metric polytope of a graph and allows to find new facet inequality in some cases. In Section 7 the notion of infinite hypermetric is briefly considered.

2. COMPUTATION OF FACETS OF HYP_8

While HYP_n is defined by an infinity of inequalities, it is proved in [11, 12, 15] that this cone is, in fact, polyhedral for all n . The proof relies on a connection with Geometry of Numbers that we now explain.

Given a quadratic form q , one can define the induced Delaunay tessellation with point set \mathbb{Z}^n ([33, 30, 20, 21]). It is well known, that in this context there are only a finite number of possible tessellations, up to the action of the group $GL_n(\mathbb{Z})$.

For a generic quadratic form, the tessellation is formed by simplices only; but, importantly, when it is not, this induces linear conditions on the coefficients of the form. There are a finite number of simplices, up to $GL_n(\mathbb{Z})$ action, and they have been classified in [18] for $n = 7$, extending previous classification for $n \leq 6$ ([28, 5, 26]). There are 11 orbits of such simplices and their volume is at most 5.

Given a simplex S , one can consider all Delaunay polytopes containing it. We consider the set Bar_S of quadratic forms, for which S is contained in a Delaunay polytope of the Delaunay tessellation. It is a polyhedral cone, called a *Baranovskii cone* in [30].

For a quadratic form inside Bar_S , the Delaunay tessellation contains S as a simplex. For a quadratic form on a facet of Bar_S , the simplex S is a part of a *repartitioning polytope*, i.e., a Delaunay polytope with $n + 2$ vertices.

If the simplex S has volume 1, then it is equivalent to the simplex formed by the vertices $v_1 = 0, v_2 = e_1, \dots, v_{n+1} = e_n$. The quadratic form q is described uniquely by the distance function $d(i, j) = q(v_i - v_j)$ on the vertices v_i . For a given positive definite quadratic form q , denote by $c(q)$ the center of the sphere, circumscribing S , and by $r(q)$ the radius of this sphere. Since S is of volume 1, a given point $v \in \mathbb{Z}^n$ can be uniquely expressed in barycentric coordinates as $v = \sum_i b_i v_i$ with $1 = \sum_i b_i$ and $b_i \in \mathbb{Z}$. In [15], the following formula is proved:

$$H(b, d) = \|v - c(q)\|^2 - r(q)^2.$$

So, the distance function d corresponds to a quadratic form q having S a part of the Delaunay tessellation if and only if it belongs to HYP_{n+1} . Therefore, in order to classify the facets of HYP_8 , we need to classify the repartitioning polytopes of dimension 7.

A repartitioning polytope R is defined by its $n + 2$ vertices w_1, \dots, w_{n+2} . There exists a unique linear relation among the vertices. It is expressed as

$$\sum_{i=1}^{n+2} \alpha_i w_i = 0 \text{ with } \sum_{i=1}^{n+2} w_i = 0.$$

It turns out, that R admits exactly two triangulations into simplices. Define

$$S_+ = \{1 \leq i \leq n + 2 \text{ s.t. } w_i > 0\} \text{ and } S_- = \{1 \leq i \leq n + 2 \text{ s.t. } w_i < 0\}.$$

The first triangulation is defined by taking all simplices with vertices $\{1, \dots, n+2\} - \{i\}$ for $i \in S_+$ and the second one similarly from S_- . Going from one triangulation to another is called a *bistellar flips* [29].

A L -type domain is the specification of the full tessellation by Delaunay polytopes of \mathbb{Z}^n . The L -type form a tessellation of the cone of positive definite quadratic forms. When one moves from one L -type to another adjacent one then the operation that is done is a change from one triangulation T to another T' . This change can be described in the following way:

- (1) Some of the simplices of the triangulation T are grouped to form repartitioning polytopes
- (2) Each repartitioning polytope has two triangulations. We go from one to another.
- (3) The triangulation T' is obtained from all the new triangulations and the simplices that did not belong to any repartitioning polytope.

As a consequence, all simplices occurring in a repartitioning polytope, are also lattice polytopes.

Now we explain our strategy for enumerating the repartitioning polytopes, which is rather similar to the one of [19]:

Theorem 1. *There are exactly 67 types of repartitioning polytopes for $n = 7$.*

Proof. Our classification is based on a computer assisted case distinction. The full list can be obtained from the web-page [17]. Here, we briefly describe the necessary ingredients and our computational steps.

Let us write the vertices of a repartitioning polytope R as v_1, \dots, v_9 . Without loss of generality, assume that v_1, \dots, v_8 are linearly independent.

For any $1 \leq i \leq n$, we define $S_i = \{v_1, \dots, v_9\} - \{v_i\}$ and denote by $\text{vol}(S_i)$ its volume multiplied by $n!$. If the points of S_i are linearly dependent, then $\text{vol}(S_i) = 0$; otherwise, S_i is a realizable simplex and so, by the classification of [18], of volume at most 5. In addition, one can assume that $\text{vol}(S_9)$ is maximal among all $\text{vol}(S_i)$ and denote it by $i(R) \leq 5$. By using exterior algebra product, one can find vectors w_i such that

$$\text{vol}(S_i) = |\langle w_i, v_9 \rangle|.$$

Since either $\text{vol}(S_i) = 0$, or $\text{vol}(S_i) \leq \text{vol}(S_9) = i(R)$, we have the inequalities:

$$-i(R) \leq \langle w_i, v_9 \rangle \leq i(R).$$

Geometrically, these conditions define a polytope, for which we are searching its integral points. One can use, for instance, the program `zsolve` from [1] to enumerate those integral points. The polytope thus defined is a little more complicated than a parallelepiped; but, as $i(R)$ increases, the condition $\text{vol}(S_9) = i(R)$ impose some strong restriction on the solution set, which, so, does not increase too much.

Hence, we obtain a finite list of possible candidates for the repartitioning polytopes. Then we need to check whether there exist an adequate quadratic form realizing it. This is done by an adaptation of Algorithm 1 of [25]. That is, we use the integral symmetries of the repartitioning polytopes and iterate until an adequate quadratic form is found. \square

After obtaining the list of 67 repartitioning polytopes, we look at all the simplices of volume 1 in it and at the corresponding barycentric coordinates of the remaining vertex. Therefore, we get:

Theorem 2. *The hypermetric cone HYP_8 has 298,592 facets in 86 orbits.*

Note that [16] gave 86 as lower bound on the number of orbits of facets, which is therefore an exact bound.

The 86 orbits O_i of facets are presented in Tables 2, 3. The first representatives of each (of 22) switching equivalence classes of facets are boldfaced there. The orbits of simplicial facets are marked by *. About 92% of the total number of facets (60 orbits) are simplicial; they polish the cone. On the other hand, each triangle facet contains about 18.8 millions of extreme rays.

For facets in HYP_n , gcd of orbit sizes is 56 if $n = 8$ and 3, 12, 10, 30, 7 if $n = 3, 4, 5, 6, 7$, respectively.

3. EXTREME RAYS OF HYP_8

A n -dimensional Delaunay polytope D is called *extreme* if up to scalar multiple there is a unique quadratic form q having D as Delaunay polytopes. For a Delaunay polytope D an *integral affine generating set* S_{aff} is a set v_1, \dots, v_m of vertices of D such that for each vertex v of D there exist $\lambda_i \in \mathbb{Z}$ such that

$$v = \sum_{i=1}^m \lambda_i v_i \text{ and } 1 = \sum_{i=1}^m \lambda_i.$$

TABLE 2. The orbits of facets of the cone HYP_8 : part 1

$F_{i,j}$	Representative	$\frac{ F_{i,j} }{56}$	CUT_8 -rank	Inc. $([0, 1], 2_{21}, 3_{21}, ER_7)$
$F_{1,1}$	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, -\mathbf{1}, \mathbf{1}, \mathbf{1})$	3	27	$(95, 329734, 737128, 17725428)$
$F_{2,1}$	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	10	27	$(79, 93978, 176058, 3780630)$
$F_{3,1}$	$(\mathbf{0}, \mathbf{0}, -\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})$	30	27	$(59, 10460, 13052, 209644)$
$F_{3,2}$	$(0, 0, -1, 1, 1, 1, 1, -2)$	15	27	$(59, 10460, 13052, 209644)$
$F_{4,1}$	$(\mathbf{0}, -\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	5	27	$(69, 36816, 60480, 1207584)$
$F_{5,1}$	$(\mathbf{0}, -\mathbf{1}, -\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3})$	15	27	$(41, 400, 240, 620)$
$F_{5,2}$	$(0, -1, 1, 1, 1, 1, 1, -3)$	6	27	$(41, 400, 240, 620)$
$F_{6,1}$	$(\mathbf{0}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, -\mathbf{2}, \mathbf{2})$	60	27	$(51, 3567, 3288, 46176)$
$F_{6,2}$	$(0, -1, -1, -1, -1, 1, 2, 2)$	15	27	$(51, 3650, 4680, 64400)$
$F_{6,3}$	$(0, 1, 1, 1, 1, 1, -2, -2)$	3	27	$(51, 3650, 4680, 64400)$
$F_{7,1}$	$(\mathbf{0}, -\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, -\mathbf{2}, \mathbf{2}, \mathbf{3})$	120	26	$(39, 311, 220, 1479)$
$F_{7,2}$	$(0, -1, -1, 1, 1, 2, 2, -3)$	90	26	$(39, 325, 172, 1461)$
$F_{7,3}$	$(0, -1, 1, 1, 1, -2, -2, 3)$	60	26	$(39, 325, 172, 1461)$
$F_{7,4}$	$(0, 1, 1, 1, 1, -2, 2, -3)$	30	26	$(39, 311, 220, 1479)$
$F_{8,1}$	$(-\mathbf{1}, -\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2})$	5	27	$(55, 6840, 8526, 141642)$
$F_{8,2}$	$(-1, -1, 1, 1, 1, 1, 1, -2)$	3	27	$(55, 6840, 8526, 141642)$
$F_{9,1}$	$(-\mathbf{1}, -\mathbf{1}, -\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{4})$	3	27	$(27, 0, 0, 0)^*$
$F_{9,2}$	$(-1, 1, 1, 1, 1, 1, 1, -4)$	1	27	$(27, 0, 0, 0)^*$
$F_{10,1}$	$(-\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, -\mathbf{2}, \mathbf{3})$	20	27	$(41, 645, 282, 3021)$
$F_{10,2}$	$(-1, -1, 1, 1, 1, 1, 2, -3)$	15	27	$(41, 645, 282, 3021)$
$F_{10,3}$	$(-1, -1, -1, -1, -1, 1, 2, 3)$	6	27	$(41, 495, 828, 5094)$
$F_{10,4}$	$(1, 1, 1, 1, 1, 1, -2, -3)$	1	27	$(41, 495, 828, 5094)$
$F_{11,1}$	$(-\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, \mathbf{1}, -\mathbf{2}, \mathbf{2}, \mathbf{2})$	30	27	$(45, 1464, 1390, 18310)$
$F_{11,2}$	$(-1, 1, 1, 1, 1, -2, -2, 2)$	15	27	$(45, 1464, 1390, 18310)$
$F_{11,3}$	$(-1, -1, -1, -1, -1, 2, 2, 2)$	1	27	$(45, 2070, 1458, 34956)$
$F_{12,1}$	$(-\mathbf{1}, -\mathbf{1}, \mathbf{1}, \mathbf{1}, -\mathbf{2}, -\mathbf{2}, \mathbf{2}, \mathbf{3})$	90	27	$(37, 293, 166, 1638)$
$F_{12,2}$	$(-1, 1, 1, 1, -2, 2, 2, -3)$	60	27	$(37, 293, 166, 1638)$
$F_{12,3}$	$(-1, -1, -1, 1, 2, 2, 2, -3)$	20	27	$(37, 306, 279, 2616)$
$F_{12,4}$	$(-1, -1, -1, -1, -2, 2, 2, 3)$	15	27	$(37, 300, 285, 2500)$
$F_{12,5}$	$(1, 1, 1, 1, -2, -2, -2, 3)$	5	27	$(37, 306, 279, 2616)$
$F_{13,1}$	$(-\mathbf{1}, -\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, -\mathbf{2}, \mathbf{2}, \mathbf{4})$	30	26	$(31, 35, 31, 31)$
$F_{13,2}$	$(-1, -1, 1, 1, 1, -2, -2, 4)$	30	26	$(31, 45, 3, 24)$
$F_{13,3}$	$(-1, -1, 1, 1, 1, 2, 2, -4)$	30	26	$(31, 45, 3, 24)$
$F_{13,4}$	$(1, 1, 1, 1, 1, -2, 2, -4)$	6	26	$(31, 35, 31, 31)$
$F_{14,1}$	$(-\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, -\mathbf{3}, \mathbf{3})$	60	26	$(35, 142, 46, 268)$
$F_{14,2}$	$(-1, 1, 1, 1, 1, -2, -3, 3)$	30	26	$(35, 142, 46, 268)$
$F_{14,3}$	$(-1, -1, -1, -1, 1, -2, 3, 3)$	15	26	$(35, 110, 142, 404)$
$F_{14,4}$	$(1, 1, 1, 1, 1, 2, -3, -3)$	3	26	$(35, 110, 142, 404)$
$F_{15,1}$	$(-\mathbf{1}, -\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, -\mathbf{3}, \mathbf{3}, \mathbf{4})$	30	26	$(26, 0, 0, 1)^*$
$F_{15,2}$	$(-1, -1, -1, 1, 1, 3, 3, -4)$	30	26	$(26, 0, 0, 1)^*$
$F_{15,3}$	$(-1, 1, 1, 1, 1, -3, -3, 4)$	15	26	$(26, 0, 0, 1)^*$
$F_{15,4}$	$(1, 1, 1, 1, 1, -3, 3, -4)$	6	26	$(26, 0, 0, 1)^*$

TABLE 3. The orbits of facets of the cone HYP_8 : part 2

$F_{i,j}$	Representative	$\frac{ F_{i,j} }{56}$	CUT_8 -rank	Inc. $([0, 1], 2_{21}, 3_{21}, ER_7)$
$F_{16,1}$	$(-1, -1, 1, -2, 2, 2, -3, 3)$	180	26	$(32, 63, 36, 177)$
$F_{16,2}$	$(1, 1, 1, -2, -2, 2, -3, 3)$	60	26	$(32, 63, 36, 177)$
$F_{16,3}$	$(-1, 1, 1, 2, 2, 2, -3, -3)$	30	26	$(32, 82, 38, 272)$
$F_{16,4}$	$(-1, 1, 1, -2, -2, -2, 3, 3)$	30	26	$(32, 82, 38, 272)$
$F_{16,5}$	$(-1, -1, -1, -2, -2, 2, 3, 3)$	30	26	$(32, 94, 26, 266)$
$F_{17,1}$	$(-1, -1, 1, 1, -2, 2, -3, 4)$	180	25	$(30, 41, 6, 51)$
$F_{17,2}$	$(-1, 1, 1, 1, -2, 2, 3, -4)$	120	25	$(30, 41, 6, 51)$
$F_{17,3}$	$(-1, -1, -1, 1, 2, 2, 3, -4)$	60	25	$(30, 40, 22, 74)$
$F_{17,4}$	$(-1, -1, -1, 1, -2, -2, 3, 4)$	60	25	$(30, 30, 32, 74)$
$F_{17,5}$	$(1, 1, 1, 1, -2, -2, -3, 4)$	15	25	$(30, 40, 22, 74)$
$F_{17,6}$	$(1, 1, 1, 1, 2, 2, -3, -4)$	15	25	$(30, 30, 32, 74)$
$F_{17,7}$	$(-1, -1, -1, -1, 2, 2, -3, 4)$	15	25	$(30, 40, 22, 72)$
$F_{18,1}$	$(-1, -1, 1, -2, 2, 3, 3, -4)$	180	25	$(27, 13, 9, 20)$
$F_{18,2}$	$(-1, 1, 1, 2, 2, -3, 3, -4)$	180	25	$(27, 15, 3, 15)$
$F_{18,3}$	$(-1, 1, 1, -2, -2, -3, 3, 4)$	180	25	$(27, 15, 3, 15)$
$F_{18,4}$	$(-1, -1, -1, -2, 2, -3, 3, 4)$	120	25	$(27, 13, 9, 18)$
$F_{18,5}$	$(-1, -1, 1, 2, 2, -3, -3, 4)$	90	25	$(27, 21, 1, 22)$
$F_{18,6}$	$(1, 1, 1, -2, 2, -3, -3, 4)$	60	25	$(27, 13, 9, 20)$
$F_{18,7}$	$(1, 1, 1, -2, -2, 3, 3, -4)$	30	25	$(27, 21, 1, 22)$
$F_{19,1}$	$(-1, -1, -1, 1, -2, -2, 2, 5)$	60	24	$(24, 2, 1, 0)^*$
$F_{19,2}$	$(-1, -1, 1, 1, 2, 2, -5)$	30	24	$(24, 3, 0, 0)^*$
$F_{19,3}$	$(-1, 1, 1, 1, -2, -2, -5)$	20	24	$(24, 3, 0, 0)^*$
$F_{19,4}$	$(1, 1, 1, 1, -2, 2, 2, -5)$	15	24	$(24, 2, 1, 0)^*$
$F_{20,1}$	$(-1, -1, -1, 1, -2, -3, 3, 5)$	120	24	$(24, 1, 1, 1)^*$
$F_{20,2}$	$(-1, -1, 1, 1, 2, -3, -3, 5)$	90	24	$(24, 2, 0, 1)^*$
$F_{20,3}$	$(-1, -1, -1, 1, 2, 3, 3, -5)$	60	24	$(24, 2, 0, 1)^*$
$F_{20,4}$	$(-1, 1, 1, 1, -2, 3, 3, -5)$	60	24	$(24, 2, 0, 1)^*$
$F_{20,5}$	$(1, 1, 1, 1, 2, -3, 3, -5)$	30	24	$(24, 1, 1, 1)^*$
$F_{20,6}$	$(1, 1, 1, 1, -2, -3, -3, 5)$	15	24	$(24, 2, 0, 1)^*$
$F_{21,1}$	$(-1, 1, -2, 2, 2, -3, -3, 5)$	180	23	$(23, 2, 1, 1)^*$
$F_{21,2}$	$(-1, -1, -2, -2, 2, -3, 3, 5)$	180	23	$(23, 3, 0, 1)^*$
$F_{21,3}$	$(-1, 1, 2, 2, 2, -3, 3, -5)$	120	23	$(23, 2, 1, 1)^*$
$F_{21,4}$	$(1, 1, -2, -2, 2, 3, 3, -5)$	90	23	$(23, 2, 1, 1)^*$
$F_{21,5}$	$(-1, -1, -2, 2, 2, 3, 3, -5)$	90	23	$(23, 3, 0, 1)^*$
$F_{21,6}$	$(1, 1, -2, -2, -2, -3, 3, 5)$	60	23	$(23, 2, 1, 1)^*$
$F_{22,1}$	$(-1, -1, 1, -2, 2, 3, 4, -5)$	360	23	$(23, 2, 1, 1)^*$
$F_{22,2}$	$(-1, -1, 1, -2, -2, -3, 4, 5)$	180	23	$(23, 2, 1, 1)^*$
$F_{22,3}$	$(-1, 1, 1, -2, -2, 3, -4, 5)$	180	23	$(23, 3, 0, 1)^*$
$F_{22,4}$	$(-1, 1, 1, 2, 2, -3, 4, -5)$	180	23	$(23, 3, 0, 1)^*$
$F_{22,5}$	$(-1, -1, 1, 2, 2, -3, -4, 5)$	180	23	$(23, 3, 0, 1)^*$
$F_{22,6}$	$(1, 1, 1, -2, 2, -3, -4, 5)$	120	23	$(23, 2, 1, 1)^*$
$F_{22,7}$	$(-1, -1, -1, -2, 2, 3, -4, 5)$	120	23	$(23, 2, 1, 1)^*$
$F_{22,8}$	$(1, 1, 1, 2, 2, 3, -4, -5)$	60	23	$(23, 2, 1, 1)^*$
$F_{22,9}$	$(1, 1, 1, -2, -2, 3, 4, -5)$	60	23	$(23, 3, 0, 1)^*$

In the special case $m = n + 1$ the set S_{aff} is called an *affine basis*. Given an extreme Delaunay polytope D of associated quadratic form q and an integral affine generating set $S_{aff} = \{v_1, \dots, v_m\}$ the distance function $d(i, j) = q(v_i - v_j)$ defines an extreme ray of HYP_m . It is proved in [15] that all extreme rays of HYP_n are obtained in this way. Thus in order to classify the extreme rays of HYP_n we need the list of extreme Delaunay polytopes of dimension at most $n - 1$ and their integral affine generating sets.

The perfect Delaunay polytopes of dimension at most 6 are determined in [13]. The ones of dimension 7 are determined in [18]. In summary, we obtain the following:

- (1) The 1-dimensional polytope $[0, 1]$. It has 4 orbits of 8-points integral affine generating sets and this yield 127 extreme rays of HYP_8 .
- (2) The 6-dimensional *Schlaflı polytope* 2_{21} . It has 195 orbits of 8 points integral affine generating sets. This gives a total of 231, 596 extreme rays in HYP_8 .
- (3) The 7-dimensional *Gosset polytope* 3_{21} . It has 374 orbits of affine basis and gives a total of 7, 126, 560 extreme rays of HYP_8 .
- (4) The 7-dimensional *Erdahl-Rybnikov polytope* ER_7 . It has 8, 430 orbits of affine basis and gives a total of 235, 337, 144 extreme rays of HYP_8 .

For $P = 3_{21}$ or ER_7 , it suffices to enumerate the orbits of 8 vertices in P . We keep the ones that determine a simplex of volume 1 and thus are integral affinely generating. We found 374 and 8, 430 orbits.

For $P = 2_{21}$, the integral affine generating sets can have 7 points with one repeated or 8 points. In the 7-points case we enumerate the affine basis of 2_{21} and consider all ways to duplicate points. In The 8-points case we enumerate all orbits of 8-points and check the ones that integrally affine generates 2_{21} . This gives 195 orbits.

For the interval $[0, 1]$, we simply have to look at the number $2^{n-1} - 1$ of non-zero cut semimetrics on n vertices.

So, altogether HYP_8 has 242, 695, 427 vertices in 9, 003 orbits.

4. HYPERMETRIC POLYTOPE

For both, the metric cone and the cut cone, there are polytope analogs. The *cut polytope* $CUTP_n$ is defined as the convex hull of all 2^{n-1} cut semimetrics. The *metric polytope* $METP_n$ is defined by the same $3\binom{n}{3}$ triangle inequalities, as MET_n , and $\binom{n}{3}$ additional *perimeter inequalities*

$$d_{ij} + d_{jk} + d_{ki} \leq 2 \text{ for } 1 \leq i < j < k \leq n.$$

The polytopes $CUTP_n$ and $METP_n$ are invariant under the following *switching* operation U_S on semimetrics:

$$U_S(d) = \begin{cases} \{1, \dots, n\}^2 & \rightarrow \mathbb{R} \\ (i, j) & \mapsto \begin{cases} 1 - d(i, j) & \text{if } |S \cap \{i, j\}| = 1 \\ d(i, j) & \text{otherwise.} \end{cases} \end{cases}$$

By analogy with above, we proceed in the following way for hypermetrics. Given a vector $b = (b_1, \dots, b_n) \in \mathbb{Z}^n$ with $\sum_{i=1}^n b_i = 2s + 1$ and $s \in \mathbb{Z}$, we define the *hypermetric polytope* $HYPP_n$ by the inequalities

$$\sum_{1 \leq i < j \leq n} b_i b_j d(i, j) \leq s(s + 1).$$

One obtains $METP_n$ using only b of the form $(1, 1, -1, 0^{n-3})$ and $(1, 1, 1, 0^{n-3})$.

Theorem 3. *Given a distance function d , there is an algorithm for testing if*

- (i) $d \in HYP_n$
- (ii) $d \in HYPP_n$

Proof. To check if $d \in HYP_n$ is to check if for all $b \in \mathbb{Z}^n$ with $1 = \sum_i b_i$ we have

$$\sum_{1 \leq i < j \leq n} b_i b_j d_{ij} \leq 0$$

while to check if $d \in HYPP_n$ is to check if for all $b \in \mathbb{Z}^n$ with $\sum_i b_i$ odd we have

$$\sum_{1 \leq i < j \leq n} b_i b_j d_{ij} \leq s(s+1) \text{ with } 2s+1 = \sum_i b_i.$$

Both questions can be reframed in terms of quadratic functions, i.e. functions that are sum of a quadratic form, linear form, and constant term. Given a quadratic function f , we need to check if there exist a point $x \in \mathbb{Z}^n$ such that $f(x) < 0$. By standard linear algebra rewriting the question becomes to check if for a positive definite quadratic form A , vector $c \in \mathbb{R}^n$ and distance d there exist a vector $x \in \mathbb{Z}^n$ such that $A[x - c] < d$. This is a Closest Vector Problem and there are algorithms for solving such questions [22]. \square

The skeletons of $HYPP_n, HYP_n$ contain a clique consisting of all cuts or all non-zero cuts. We expect that any vertex is adjacent to a cut vertex (it holds for $n \leq 8$); if true, it will imply that each of above skeletons have diameter 3. The *ridge graphs* (i.e., skeletons of the duals) of the triangle/perimeter facets of $METP_n, MET_n$ with $n \geq 4$ have diameter 2 ([8]). We expect that any facet of $HYPP_n, HYP_n$ is adjacent to a triangle/perimeter facet (it holds for $n \leq 7$); if true, it will imply that the ridge graphs of $HYPP_n, HYP_n$ have diameter 4.

Theorem 4. *The entries of Table 1 for the hypermetric polytopes $HYPP_7$ and $HYPP_8$ are valid.*

Proof. The hypermetric cone HYP_8 has 9,003 orbits of extreme rays and 86 orbits of facets under $Sym(8)$. We consider the group of order $2^7 8!$, denote it by $Ares(K_8)$, generated by $Sym(8)$ and the switchings. Under the group $Ares(K_8)$ the facets of HYP_8 generate 22 orbits of facets of $HYPP_8$. The extreme rays of HYP_8 are of the form λv with v a generator. For each extreme ray, we choose λ to be the maximal value which defines a vertex of the hypermetric polytope $HYPP_8$ by using Theorem 3. After elimination of isomorphic pairs, this gives 581 orbits of vertices of $HYPP_8$.

However, there could be more facets of $HYPP_8$: In principle it could happen that a vector (b_1, \dots, b_n) is not incident to any cut and yet defines a facet of $HYPP_8$. There could be more vertices as well: If a vertex of $HYPP_8$ is not adjacent to any cut, then it does not appear from the list of extreme rays of HYP_8 . For each of the 581 orbits of vertices we compute the adjacent vertices by using the list of 22 orbits that we have. All the vertices found belong to the 581 orbits, which proves that both lists are complete.

The same method applies as well to $HYPP_7$. \square

The hypermetric polytope $HYPP_8$ has 581 orbits of vertices, which are in details:

- 1 orbit V_C of 128 cuts. The stabilizer of a cut is isomorphic to $Sym(8)$; the number of classes is 4. V_C forms a clique; the cone of facets incident to a

TABLE 4. Orbits of facets of the hypermetric polytope HYP_8

F_i	Representative	$\frac{ F_i }{32}$	#classes	Inc. $([0, 1], \{2_{21}, 3_{21}\}, ER_7)$
F_1	(0, 0, 0, 0, 0, 1, 1, 1)	7	2	(96, 1598784, 80836608)
F_2	(0, 0, 0, 1, 1, 1, 1, 1)	28	3	(80, 383040, 14300640)
F_3	(0, 1, 1, 1, 1, 1, 1, 1)	16	4	(70, 131712, 3975552)
F_4	(0, 0, 1, 1, 1, 1, 1, 2)	168	6	(60, 32160, 590960)
F_5	(0, 1, 1, 1, 1, 1, 2, 2)	336	9	(52, 9600, 122160)
F_6	(1, 1, 1, 1, 1, 1, 1, 2)	32	8	(56, 19656, 370272)
F_7	(0, 1, 1, 1, 1, 1, 1, 3)	112	7	(42, 840, 1120)
F_8	(1, 1, 1, 1, 1, 2, 2, 2)	224	12	(46, 3528, 39906)
F_9	(0, 1, 1, 1, 1, 2, 2, 3)	1,680	15	(40, 656, 2686)
F_{10}	(1, 1, 1, 1, 1, 1, 2, 3)	224	14	(42, 1323, 6489)
F_{11}	(1, 1, 1, 1, 1, 1, 1, 4)	32	8	(28, 0, 0)*
F_{12}	(1, 1, 1, 1, 1, 2, 3, 3)	672	18	(36, 252, 464)
F_{13}	(1, 1, 1, 1, 2, 2, 2, 3)	1,120	20	(38, 585, 3210)
F_{14}	(1, 1, 1, 1, 1, 2, 2, 4)	672	18	(32, 66, 36)
F_{15}	(1, 1, 1, 2, 2, 2, 3, 3)	2,240	24	(33, 120, 302)
F_{16}	(1, 1, 1, 1, 2, 2, 3, 4)	3,360	30	(31, 62, 82)
F_{17}	(1, 1, 1, 1, 2, 2, 2, 5)	1,120	20	(25, 3, 0)*
F_{18}	(1, 1, 1, 1, 1, 3, 3, 4)	672	18	(27, 0, 1)*
F_{19}	(1, 1, 1, 2, 2, 3, 3, 4)	6,720	36	(28, 22, 22)
F_{20}	(1, 1, 1, 1, 2, 3, 3, 5)	3,360	30	(25, 2, 1)*
F_{21}	(1, 1, 2, 2, 2, 3, 3, 5)	6,720	36	(24, 3, 1)*
F_{22}	(1, 1, 1, 2, 2, 3, 4, 5)	13,440	48	(24, 3, 1)*

cut is exactly the hypermetric cone HYP_8 . Cuts are the only vertices of HYP_n , having all coordinates integral.

- 24 orbits corresponding to the Delaunay polytopes 2_{21} and 3_{21} . Details on those orbits are given in Table 5. The denominator of the coordinates is 3 for all vertices.
- 556 orbits corresponding the extreme Delaunay polytope ER_7 . The denominator of the coordinates is 12 for all vertices. The number of incident inequalities and adjacent vertices is 28 for each of them.

5. STRUCTURE FOR ALL SIMPLICES

The hypermetric cone HYP_{n+1} describes the possible ways, in which the simplex S on vertices $0, e_1, \dots, e_n$ can be embedded in a Delaunay polytope. In particular, we saw in Section 2 how the facets of HYP_{n+1} correspond to the repartitioning polytopes, in which S can be embedded.

In dimension $n \geq 5$ there are other simplices, necessarily of volume higher than 1, that can define Delaunay polytopes. In [3], it was determined that the possible volumes of 5-dimensional Delaunay simplices are 1 and 2. In [4, 6], it was determined that the volumes of 6-dimensional Delaunay simplices are 1, 2 and 3. Then, in [27], the facets of the corresponding Baranovskii cones Bar_S were found.

TABLE 5. Orbits of vertices of the hypermetric polytope $HYPP_8$ originating from extreme Delaunay polytopes 2_{21} and 3_{21} . Column 2 is the order of the stabilizer; column 5 is number of orbits of type 2_{21} and 3_{21} that merged into single orbit V_i in $HYPP_8$; columns 6, 7 are the number of facets, containing the orbit representative, and the number of vertices of $HYPP_8$ adjacent to it

V_i	$ Stab $	$\frac{ V_i }{10.752}$	$\#orbits\ Sym(8)$	Merging 23_{21}	Incidence	Adjacency
V_1	24	20	36	$2_{21}(8), 3_{21}(12)$	112	848
V_2	48	10	30	$2_{21}(7), 3_{21}(10)$	104	799
V_3	96	5	23	$2_{21}(5), 3_{21}(8)$	94	701
V_4	12	40	48	$2_{21}(11), 3_{21}(16)$	94	758
V_5	8	60	46	$2_{21}(10), 3_{21}(16)$	94	804
V_6	12	40	40	$2_{21}(8), 3_{21}(16)$	92	979
V_7	240	2	18	$2_{21}(2), 3_{21}(5)$	86	926
V_8	12	40	48	$2_{21}(11), 3_{21}(16)$	86	709
V_9	8	60	54	$2_{21}(12), 3_{21}(18)$	86	728
V_{10}	4	120	72	$2_{21}(16), 3_{21}(24)$	86	774
V_{11}	16	30	33	$2_{21}(6), 3_{21}(13)$	84	1,070
V_{12}	4	120	60	$2_{21}(12), 3_{21}(24)$	84	963
V_{13}	48	10	26	$2_{21}(4), 3_{21}(8)$	82	1,023
V_{14}	12	40	48	$2_{21}(7), 3_{21}(18)$	81	1,080
V_{15}	4	120	60	$2_{21}(9), 3_{21}(30)$	79	935
V_{16}	20	24	24	$2_{21}(5), 3_{21}(8)$	78	734
V_{17}	16	30	33	$2_{21}(7), 3_{21}(12)$	78	679
V_{18}	8	60	46	$2_{21}(10), 3_{21}(16)$	78	690
V_{19}	4	120	56	$2_{21}(12), 3_{21}(20)$	78	716
V_{20}	4	120	56	$2_{21}(9), 3_{21}(25)$	78	1,050
V_{21}	60	8	16	$2_{21}(2), 3_{21}(6)$	76	1,070
V_{22}	4	120	48	$2_{21}(9), 3_{21}(20)$	76	941
V_{23}	16	30	29	$2_{21}(5), 3_{21}(11)$	74	1,032
V_{24}	4	120	48	$2_{21}(8), 3_{21}(22)$	74	920

TABLE 6. Incidence between the orbits V_i of vertices and the orbits F_j of facets of $HYPP_7$. For each representative of orbit V_i , the number of incident facets is given.

	F_1	F_2	F_3	F_4	F_5	F_6	F_7
V_1	105	210	35	630	546	147	2100
V_2	8	6	0	4	2	0	1
V_3	11	6	1	2	1	0	0
V_4	12	7	0	2	0	0	0
V_5	15	5	1	0	0	0	0
V_6	14	7	0	0	0	0	0

TABLE 7. All types of Delaunay simplices in \mathbb{R}^7 , their volume, size of automorphism group and the number of facets of the cones Bar_S

S_i	$\text{vol}(S_i)$	$ Aut $	$\#facets\ Bar_{S_i}$
S_1	1	40,320	298,592(86)
S_2	2	40,320	5,768(9)
S_3	2	1,440	6,590(62)
S_4	3	540	966(9)
S_5	3	1,152	728(9)
S_6	3	240	640(39)
S_7	4	1,440	28(3)
S_8	4	240	153(11)
S_9	4	144	131(10)
S_{10}	5	72	28(6)
S_{11}	5	48	28(8)

In [18], the list of all possible 7-dimensional Delaunay simplices has been determined and from that we can get for each simplex its associated Baranovskii cone Bar_S . There are 11 types of such simplices and, in contrast to the lower dimensional cases, two simplices can have the same volume and yet be inequivalent. Key information about those Delaunay simplices are given in Table 7.

6. HYPERMETRICS ON GRAPHS

Given a graph $G=(V,E)$, the notion of cut is well defined. It suffices to restrict the cut semimetric on the edges of the graph and one obtains the *cut polytope of the graph* $CUTP(G)$. The notion of $METP_n$ can also be extended to the graph setting but requires more work: for a cycle C and an odd sized set F of edges in C , the *cycle inequality* $m_{C,F}$ is defined as

$$\sum_{e \in C-F} x_e - \sum_{e \in F} x_e \leq |F| - 1.$$

The *metric polytope of the graph* $METP(G)$ is defined as the polytope defined by all cycle inequalities $m_{C,F}$ and the *non-negativity inequalities* $x_e \in [0, 1]$. In fact, it is the projection of MET_n on $\mathbb{R}^{|E|}$, indexed by the edges of G . It is known that $METP(G) = CUTP(G)$ if and only if G has no K_5 -minor.

Proposition 1. *Let us take a valid inequality on $CUTP_n$ of the form*

$$f(x) = \sum_{1 \leq i < j \leq n} a_{ij} x_{ij} \leq C.$$

Suppose that we have n vertices v_1, \dots, v_n of G with any two vertices v_i, v_j being joined by a such path P_{ij} that:

- *the edge set of all paths P_{ij} are disjoint;*
- *if $a_{ij} > 0$, then P_{ij} is reduced to an edge.*

Then, the following inequality

$$i_{f,G}(x) = \sum_{1 \leq i < j \leq n} a_{ij} \left(\sum_{e \in P_{ij}} x_e \right) \leq C$$

is valid on $CUTP(G)$.

Proof. Let us take a cut of $G = (V, E)$ defined by $S \subset V$. If S cuts the paths P_{ij} in at most one edge, then the inequality on $CUTP(G)$ reduces to the one on $CUTP_n$ and so, is valid.

In the general case, we will create a new cuts S' , which will allow us to prove the required inequality. If S cuts P_{ij} in more than one edge, then $a_{ij} \leq 0$. If both v_i and v_j are in the same part of the partition $(S, V - S)$, then we set all vertices of P_{ij} to be in the same part of the partition $(S', V - S')$. Otherwise, there exists an edge $e = \{w_i, w_j\} \in P_{ij}$ cut by S . If $w_j \in S'$, then we set the vertices between w_j and v_j to belong to S' and we set the vertices from w_i to v_i not to belong to it. Due to the sign condition on a_{ij} , one obtains

$$i_{f,G}(\delta_S) \leq i_{f,G}(\delta_{S'}).$$

Then, from S' we can obtain very simply a cut S'' on K_n and this gives

$$i_{f,G}(\delta_{S'}) = f(\delta_{S''}) \leq C,$$

which proves the required inequality. \square

When applied to the metric inequalities of K_n and taking switchings, the above proposition gives us the metric polytope $METP(G)$. Therefore, it is tempting to define the hypermetric polytope $HYPP(G)$ as the polytope defined by the switchings of the extension of of all hypermetric inequalities obtained from above Proposition. What is not clear is when $CUTP(G) = HYPP(G)$ and whether there is a nice characterization of such hypermetrics. Above discussion is applied also to cones.

Any K_n subgraph of G will satisfy the hypothesis and the facets of $CUTP_n$ will give facets of $CUTP(G)$. Proposition 1 gives valid inequality induced by a class of homeomorphic K_n . (A graph H is *homeomorphic* to a subgraph of G if H can be mapped to G so that the edges of H are mapped to disjoint paths in G .) A graph H is a *minor* of G if H can be obtained from G by deleting edges and vertices and contracting edges. An homeomorphic graph K_n is a special case of a K_n minor. In [31] it is proved that $CUTP(G) = METP(G)$ if and only if G has no K_5 as a minor. However, the proof appears nonconstructive and does not seem to be able to give hypermetric inequalities, or their generalization, in a straightforward way.

7. INFINITE HYPERMETRICS

Another interesting question is to define infinite hypermetric cones. One way to do that is to define $HYPP_\infty$ by imposing that for all $b \in \mathbb{Z}^\infty$ with finite *support* (i.e., the set $\{i : b_i \neq 0\}$) and $\sum_i b_i = 1$ it holds

$$\sum_{i < j} b_i b_j d_{ij} \leq 0.$$

For example, the path metric of the skeleton of the infinite hyperoctahedron $K_{2,\dots,2,\dots}$ is an infinite hypermetric, which does not embed isometrically into l_1 . In general, it is easy to build infinite hypermetrics; it basically suffices to use Delaunay

polytopes of infinite lattices. For example, above ∞ -dimensional hyperoctahedron can be considered a Delaunay polytope in the ∞ -dimensional root lattice D_∞ .

However, as far as we know, there is no general theory of Delaunay polytopes in infinite dimensional lattices.

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